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A Class of Non-embeddable Designs

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A sufficient condition for non-embeddability of quasi-derived designs into symmetric designs is proved. Using this condition, several series of non-embeddable designs are constructed.

1. INTRODUCTION

We assume that the reader is familiar with the basic notions and facts from design theory. Our notation follows that from [1–3, 11].

Given a symmetric (v, k, λ) design D and a block B in D, removing B and all its points from the remaining blocks yields a $2-(v-k, k-\lambda, \lambda)$ design called *residual* (with respect to B). Similarly, the points of B and the intersections of B with the remaining blocks form a $2-(k, \lambda, \lambda-1)$ design called *derived.* A $2-(v, k, \lambda)$ design with $r=k+\lambda$ is thus called *quasi-residual*, and a $2-(v, k, \lambda)$ design with $\lambda=k-1$ is *quasi-derived.*

A natural question that arises is whether a given quasi-residual or quasi-derived design is embeddable as residual or derived design into a corresponding symmetric design. Replacing a symmetric design by its complementary design transforms its residual designs into derived designs and vice versa. Thus the notions "quasi-residual" and "quasi-derived" are not essentially different. Still, there is a difference between the problems of embedding of quasi-residual and quasi-derived designs due to the following obvious reasons. The usual assumption $k<v/2$ for a symmetric design

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implies that a residual design contains more information than a derived one, since the former contains more than half of the points. On the other hand, in a quasi-residual $2-(v, k, \lambda)$ design with $k < v/2$ it may happen that some pair of blocks meet in more than λ points, which implies immediately non-embeddability. Of this type is, for instance, the first non-trivial example of a non-embeddable $2-(16, 6, 3)$ design found by Bhattacharya. Evidently, this situation never occurs in the case of a quasi-derived design, and neither for a quasi-residual design with $k \geqslant v/2$.

Another even more trivial reason for non-embeddability can be the non-existence of a corresponding symmetric design by the Bruck–Ryser–Chowla theorem. In fact, most of the known examples of non-embeddable quasi-derived designs with $k < v/2$ (or equivalently, quasi-residual designs with $k > v/2$) are of this type.

In [5] we constructed a class of quasi-residual designs with $k > v/2$ containing non-embeddable designs even though symmetric designs with the related parameters might exist. The designs from [5] possessed certain subdesigns and we used ideas from coding theory to show that these designs are not embeddable. In the present paper we show that under some extra conditions of combinatorial nature, the subdesign already makes the design non-embeddable. More precisely, we construct non-embeddable designs with the following parameters: $2-(n^2+1, n, n-1)$ designs for $n > 2$ a prime power, $2-(n^2, n, n-1)$ designs for n^2-n-1 a prime power, $n > 2$, and $2-(n^2(n^2-n-1)+1,\ n^2-n, n^2-n-1)$ designs for $n \geqslant 3$ and $n^2(n^2-n-2)+1$ both prime powers.

2. A Sufficient Condition for Non-embeddability

THEOREM 2.1. *Let D be a $2-(v, k, k-1)$ design containing a $2-(v_0, k, k-1)$ subdesign D_0. A necessary condition for D to be embeddable as a derived design into a symmetric $2-(v(v-1)/k+1, v, k)$ design is the following inequality*:

$$v_0(v_0-1)[(v_0-1)(v_0-k)-2(v-k)]+2(v-1)(v-k) \geqslant 0. \qquad (2.1)$$

Proof. Denote by b (resp. b_0) the total number of blocks in D (resp. D_0). The b_0 blocks of the $2-(v_0, k, k-1)$ subdesign D_0 have to be extended by $(v-k)$-subsets of a set of $b+1-v$ new points in such a way that any two of the extended blocks meet in precisely k points. Therefore, the dual of this structure is a pairwise balanced design D' with parameters $v' = b_0$, $r' = v$, $\lambda' = k$, $b' = b+1$. The points of D_0 define v_0 blocks of D' of size $v_0 - 1$, while the remaining $v - v_0$ points of D correspond to empty blocks in D'. For the remaining $b+1-v$ blocks of D', let us denote by n_i the number of blocks of size i. We have

$$\sum n_i = b+1-v,$$

$$\sum in_i = b_0 v - v_0(v_0-1),$$

$$\sum i(i-1)\, n_i = b_0(b_0-1)k - v_0(v_0-1)(v_0-2).$$

Evidently

$$\sum (i-1)(i-2)\, n_i \geqslant 0.$$

On the other hand,

$$\begin{aligned}\sum (i-1)(i-2)\, n_i &= \sum i(i-1)\, n_i - 2\sum in_i + 2\sum n_i\\ &= b_0(b_0-1)k - v_0(v_0-1)(v_0-2)\\ &\quad - 2[b_0 v - v_0(v_0-1)] + 2(b+1-v),\end{aligned}$$

from which the inequality (2.1) follows after substituting $b = v(v-1)/k$ and $b_0 = v_0(v_0-1)/k$. ∎

Taking $v_0 = k+1$, one obtains the following theorem:

THEOREM 2.2. *Let D be a $2-(v, k, k-1)$ design containing a $2-(k+1, k, k-1)$ subdesign D_0. Then D is not embeddable as a derived design into a symmetric $2-(v(v-1)/k+1, v, k)$ design provided that*

$$(k+1)(v-k) - (k+1)\, k/2 > (v-1)(v-k)/k. \tag{2.2}$$

Non-embeddable $2-(v, k, k-1)$ designs containing a $2-(v_0, k, k-1)$ subdesign with the additional property that all blocks meet the point set of the subdesign, i.e., the subdesign is a blocking set, were constructed in our earlier paper [5]. Under some extra assumptions on the parity of v and k it was possible to prove the non-embeddability by simple arguments from coding theory.

In the next section we use the criterion of Theorem 2.1 or 2.2 to construct some classes of non-embeddable $2-(v, k, k-1)$ designs with a subdesign that is not always a blocking set, and not necessarily the complete 2-$(k+1, k, k-1)$ design.

3. Some Classes of Non-embeddable Quasi-derived Designs

Lemma 3.1. *Suppose that there exists a* $2-(n^2-n-1, n-1, n-1)$ *design* D $(n>2)$ *such that the block set of* D *can be partitioned into* $n+1$ *classes of size* n^2-n-1 *in such a way that each point occurs precisely* $n-1$ *times in each class; in other words,* D *is* $(n-1)$*-resolvable. Then* D *is embeddable into a* $2-(n^2, n, n-1))$ *design* D^*.

Proof. Construct a design D^* as follows. Extend the point set of D with $n+1$ new points. Adjoin one new point to all blocks from a given class of the block partition of D, different classes being enlarged by distinct new points. Finally, add $n+1$ new blocks of size n forming the trivial $2-(n+1, n, n-1)$ design on the set of the new points. ▮

Lemma 3.2. *The* $2-(n^2, n, n-1)$ *design* D^* *constructed in Lemma* 3.1 *is not embeddable as a derived design into a symmetric* $2-(n^3-n+1, n^2, n)$ *design if* $n>2$.

Proof. Apply Theorem 2.2. ▮

At first glance, the conditions under which the construction of Lemma 3.1 works may seem very restrictive. However, the next theorem shows that designs of this type are not so rare.

Theorem 3.3. *If* $n^2-n-1=q$ *is a prime power,* $n>2$, *then there exists a non-embeddable* $2-(n^2, n, n-1)$ *design* D^*.

Proof. A $2-(n^2-n-1, n-1, n-1)$ design D with the required block partition as in Lemma 3.1 can be constructed as follows. Take as point set $GF(q)$. Since $q-1=n^2-n-2=(n+1)(n-2)$, the multiplicative group of $GF(q)$ has a subgroup H of order $n-2$. Let $B=HU\{0\}$. The orbit of B under the doubly transitive group $GA(q)$ of affine transformations of $GF(q)$ is a 2-design D with q points, block size $n-1$, and

$$b=|GA(q)|/(n-1)=q(q-1)/(n-1)=(n^2-n-1)(n+1)$$

blocks (cf., e.g., [11, 1.6]). Hence, D is a $2-(n^2-n-1, n-1, n-1)$ design. Moreover, the additive group of $GF(q)$ provides the required partition of the blocks into $n+1$ classes of size $q=n^2-n-1$. ▮

Remark 1. (i) The designs for $n=3$ $(2-(9, 3, 2))$ and $n=5$ $(2-(25, 5, 4))$ appeared already in [5], although their non-embeddability was proved there by use of codes. There are 36 non-isomorphic $2-(9, 3, 2)$ designs [6, 8]. Precisely six of those are non-embeddable: five designs do contain $2-(4, 3, 2)$ sybdesigns, and the sixth (No. 6 in [6]) does not;

we note that it was incorrectly claimed in [9, p. 194] that all six non-embeddable $2-(9,3,2)$ designs contain $2-(4,3,2)$ subdesigns.

(ii) For $n=4$ we obtain a non-embeddable solution for the parameter set No. 12 from the table in [5].

Table I lists the first few parameters of designs constructed in Theorem 3.3. Question is whether there are infinitely many values of n for which n^2-n-1 is a prime power.

Remark 2. Dieter Jungnickel [4] has investigated the smallest possible λ for a quasi-multiple of an affine or projective plane, i.e., the smallest $\lambda=a(n)$ (resp. $\lambda=b(n)$) for which a $2-(n^2, n, a(n))$ (resp. a $2-(n^2+n+1, n+1, b(n))$) design can exist. Clearly $a(n)=b(n)=1$ if n is a prime power. The designs constructed in Theorem 3.3 show that $a(n) \leqslant n-1$ provided that n^2-n-1 is a prime power. Table II gives values of $n<100$, for which the designs from Theorem 3.3 have smaller number of blocks than previously known. We are thankful to Dieter Jungnickel for providing us with the data of Table II.

Remark 3. The designs from Theorem 3.3 can also be considered as generalized quasi-residual designs for generalized symmetric $2-(n^2+n+1, n+1, n-1)$ designs (cf. [10, 7]). A necessary and sufficient condition for

TABLE I

Non-embeddable $2-(n^2, n, n-1)$ Designs

n	n^2-n-1	$2-(n^2-n-1, n-1, n-1)$	$2-(n^2, n, n-1)$
3	5	$2-(5,2,2)$	$2-(9,3,2)$
4	11	$2-(11,3,3)$	$2-(16,4,3)$
5	19	$2-(19,4,4)$	$2-(25,5,4)$
6	29	$2-(29,5,5)$	$2-(36,6,5)$
7	41	$2-(41,6,6)$	$2-(49,7,6)$
8	$55=5.11$	?	?
9	71	$2-(71,8,8)$	$2-(81,9,8)$
10	89	$2-(89,9,9)$	$2-(100,10,9)$
11	109	$2-(109,10,10)$	$2-(121,11,10)$
12	131	$2-(131,11,11)$	$2-(144,12,11)$
13	$155=5.31$	?	?
14	181	$2-(181,13,13)$	$2-(196,14,13)$
15	$209=11.19$	?	?
16	239	$2-(239,15,15)$	$2-(256,16,15)$
17	271	$2-(271,16,16)$	$2-(279,17,16)$
18	$305=5.61$	?	?
19	$341=11.31$	?	?
20	379	$2-(379,19,19)$	$2-(400,20,19)$

TABLE II

Small Quasi-multiples of Affine Planes $2-(n^2, n, \lambda)$, $n < 100$

n	n^2-n-1	New λ (Theorem 3.3)	Old λ [4]
21	419	20	21
22	461	21	24
36	1259	35	36
39	1481	38	64
45	1979	44	60
46	2069	45	48
51	2549	50	72
54	2861	53	56
55	2969	54	176
56	3079	55	56
57	3191	56	57
66	4289	65	8976
69	4691	68	96
77	5851	76	154
86	7309	85	88
87	7481	86	120
94	8741	93	96
95	8929	94	380

embeddability of a $2-(n^2, n, n-1)$ design D as a generalized residual into a generalized symmetric $2-(n^2+n+1, n+1, n-1)$ design D' is that D is $(n-1)$-resolvable [10]. In such a case, adding a new point to all blocks from a given class and adding $n-1$ identical blocks consisting of all $n+1$ new points, one obtains a corresponding generalized symmetric design D'.

In this respect we would like to ask the following question:

Question. Are the designs from Theorem 3.3 residuals of generalized symmetric designs?

Since all $2-(9, 3, 2)$ designs are 2-resolvable [8, 6], the answer is "yes" in the smallest case ($n=3$).

In [5] we proved by use of codes that a $2-(2n+1, 3, 2)$ design with a $2-(n, 3, 2)$ subdesign ($n \equiv 0$ or $1 \pmod 3$) is non-embeddable if n is even. Note that a $2-(n, 3, 2)$ subdesign of a $2-(2n+1, 3, 2)$ design is always a blocking set. For these parameters, inequality (2.1) gives non-embeddability only for $n=4$, in which case the subdesign is the trivial $2-(4, 3, 2)$ design.

Now we construct a class of designs with subdesigns being complements of affine planes.

THEOREM 3.4. *If* $q=n^2(n^2-n-2)+1$ *and* $n\geqslant 3$ *are both prime powers, then there exists a non-embeddable* $2-(n^2(n^2-n-1)+1,\ n^2-n,\ n^2-n-1)$ *design* D.

Proof. We construct D as a design with a $2-(n^2, n^2-n,\ n^2-n-1)$ subdesign D_0 being the complement of an affine plane of order n. Every block of D meets the point set of D_0 in either one point, or n^2-n points forming a block of D_0. The points of D which are not points of D_0 form a subdesign D_1 with parameters $2-(n^2(n^2-n-2)+1,\ n^2-n-1,\ n^2-n-1)$ which is (n^2-n-1)-resolvable. D_1 is constructed with the help of the group $GA(q)$ of affine transformations of $GF(q)$, $q=n^2(n^2-n-2)+1$ as in Theorem 3.3. The blocks of D_1 form an orbit under $GA(q)$ of $HU\{0\}$, where H is a subgroup of order n^2-n-2 of the multiplicative group of $GF(q)$. The (n^2-n-1)-resolution is provided by the action of the additive group of $GF(q)$ on the blocks, and every class of blocks from the resolution is extended with one point of D_0.

Inequality (2.1) now shows that D is non-embeddable if $n>2$. ∎

It is not clear now often n and $q=n^2(n^2-n-2)+1$ can both be prime powers. The smallest two cases are $n=3$ $(q=37)$ and $n=8$ $(q=3457)$, yielding a $2-(46, 6, 5)$ and a $2-(3521, 56, 55)$ design respectively.

The designs described in the above theorems as well as the designs from [5] all possess subdesigns that are blocking sets. Now we give a series of non-embeddable designs with subdesigns that are not blocking sets.

LEMMA 3.5. *Suppose that a group-divisible* $(n-1)$*-resolvable design* D_1 *with parameters* $v=n^2-n$, $k=n-1$, $r=n(n-1)$, $b=n^2(n-1)$, *group size* n, $\lambda_1=0$, $\lambda_2=n-1$ *exists. Then there exists a* $2-(n^2+1, n, n-1)$ *design* D *which is non-embeddable if* $n\geqslant 3$.

Proof. We construct D in the following way. Take as a subdesign of D the trivial symmetric $2-(n+1, n, n-1)$ design D_0. A point of D_0 is contained in n blocks of D_0 and n^2-n further blocks of D. Pairs of points of D_0 occur only in blocks of D_0. Therefore, the points of D_0 define a partition of the remaining $b-(n+1)=n^3-1$ blocks of D into $n+1$ classes of size n^2-n, each class consisting of all blocks not belonging to D_0 but containing a given point of D_0, and one class of $n-1$ blocks disjoint from D_0. Take the points of the given group-divisible design D_1 as the n^2-n points of D distinct from those of D_0. Since D_1 is $(n-1)$-resolvable, the block set of D_1 is partitioned into n classes of size $n(n-1)$ so that each point occurs precisely $n-1$ times in any class. Given a class C of that partition, extend all blocks from C with a given point of D_0, different classes being extended by distinct points of D_0. The n^2-n blocks of D containing the remaining point of D_0 are partitioned into $n-1$ classes, the restriction of each class being a $2-(n, n-1, n-2)$ design on a unique point group of D_1, different classes corresponding to different groups. Finally, add $n-1$ disjoint blocks each consisting of all points from a given point group of D_1. It is readily seen that D is a $2-(n^2+1, n, n-1)$ design. If $n>2$, this design is non-embeddable by Theorem 2.2. ∎

THEOREM 3.6. *If $n>2$ is a prime power, then there exists a non-embeddable $2-(n^2+1, n, n-1)$ design.*

Proof. Use Lemma 3.5 and take as D_1 a group-divisible design obtained from the affine plane $AG(2, n)$ in the following way: remove one parallel class of lines and the points of one block from that class; take $n-1$ copies of the remaining design. ∎

EXAMPLE. For $n=3$ the above construction produces the following non-embeddable $2-(10, 3, 2)$ design giving a solution for the parameter set No. 2 in [5]:

1	1	1		1	1	1	1	1	1																				
1	1		1							1	1	1	1	1	1														
1		1	1													1	1	1	1	1	1								
	1	1	1																			1	1	1	1	1	1		
				1			1			1			1			1			1			1	1					1	
					1			1			1			1			1			1		1		1				1	
						1			1			1			1			1			1		1	1				1	
				1			1				1			1				1			1				1	1			1
					1			1				1			1	1			1						1		1		1
						1			1	1			1				1			1						1	1		1.

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